

SOLID WEAK BCC-ALGEBRAS

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ABSTRACT. We characterize weak BCC-algebras in which the identity $(xy)z = (xz)y$ is satisfied only in the case when elements x, y belong to the same branch.

1. INTRODUCTION

A goal that artificial intelligence has been trying to perfect for decades is to realistically simulate decision making process of humans when faced with both certain and uncertain types of information. Logic behind those decisions is dominant in proof theory. Such logic has two important roles both in mathematics and computers – it is a technique responsible for foundations of any system as well as a tool for applications. Additionally to classical logic there are various non-classical logic systems built on top of it (e.g. many-valued logic, fuzzy logic, etc.) that handle information with various aspects of uncertainty (cf. [22]) – fuzziness, randomness and so on. Incomparability is one important example of such uncertainty that may be easily understood based on one’s experience. Computer science relies heavily on non-classical logic to deal with fuzzy and uncertain information.

Because of advancements made in technology and artificial intelligence, the study of t -norm-based logic systems and the corresponding pseudo-logic systems has become a big focus in the field of logic. BCK and BCI-algebras have been inspired by some implicational logic and that can be even noticed when looking at the similarity of names. Examples may be BCK-algebras and a BCK positive logic or BCI-algebras and a BCI positive logic. t -norm-based algebraic investigations were indeed first to the corresponding algebraic investigations and in the case of pseudo-logic systems, algebraic development was first to the corresponding logical development (see for example [14]). However, the link between such algebras and their respective logics may be even much stronger. It is possible to define translation procedures (and sometimes even their inverse procedures) for all well formed formulas and all theorems of a given logic into terms and theorems of the corresponding algebra [19]. Even when the full inverse translation procedure is impossible, many researchers find it interesting and useful for corresponding logics to study the algebras motivated by known logics (cf. [1]).

BCC-algebras (introduced by Y. Komori [16]) are generalizations of BCK-algebras, weak BCC-algebras are generalizations of BCI-algebras. In view of strong connections with a BIK^+ -logic, (weak) BCC-algebras are also called (weak) BIK^+ -algebras (cf. [24]). From results proved in [7], [8] and [18] it follows that MV-algebras are equivalent to the bounded commutative weak BCC-algebras BCK-algebras, and so

on. Hence, most of the algebras related to the t -norm based logic, such as MTL-algebras, BL-algebras, lattice implication algebras, Hilbert algebras and Boolean algebras, are special cases of weak BCC-algebras. This shows that BCC-algebras are considerably general structures.

Equivalence relations play an important role in the investigation of such algebras and the corresponding logic. Equivalence relations are determined by subsets called ideals. A special case of ideals are branches induced by some elements. It is well known that some properties which are satisfied in the ideals can be extended to the whole algebra. Thus, it is sufficient to examine these properties only on those ideals (branches).

The identity $(xy)z = (xz)y$ plays a crucial role in the study of such algebras. It holds in all BCK-algebras and in some generalizations of BCK-algebras, but not in BCC-algebras. BCC-algebras satisfying this identity are BCK-algebras (cf. [7] or [8]). The proofs of many properties of BCK, BCI and BCC-algebra are required to meet the equality of all elements of the algebra, but there are examples of algebras having the property and not satisfying this identity. Therefore, it is important to examine for which of the properties of weak BCC-algebras it is sufficient that the equation $(xy)z = (xz)y$ is satisfied by elements belonging to the same branch. This will enable faster checking whether a given algebra has this property.

Below we begin the study of weak BCC-algebras in which the above identity is satisfied only in the case when elements x and y belong to the same branch.

2. PRELIMINARIES

Definition 2.1. A *weak BCC-algebra* is a system $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms:

- (i) $((x * y) * (z * y)) * (x * z) = 0$,
- (ii) $x * x = 0$,
- (iii) $x * 0 = x$,
- (iv) $x * y = y * x = 0 \implies x = y$.

By many mathematicians, especially from China and Korea, weak BCC-algebras are called *BZ-algebras* (cf. [27], [25] or [12]), but we keep the first name because it coincides with the general concept of names presented in the book [14] for algebras of logic.

A weak BCC-algebra satisfying the identity

$$(v) \quad 0 * x = 0$$

is called a *BCC-algebra* or *BIK⁺-algebra*.

A weak BCC-algebra satisfying the identity

$$(vi) \quad (x * y) * z = (x * z) * y$$

is called a BCI-algebra. A weak BCC-algebra which is neither a BCI-algebra or a BCC-algebra is called *proper*. Proper weak BCC-algebras have at least four elements (see [9]). Note that there are only two non-isomorphic weak BCC-algebras with four elements:

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 2 | 2 |
| 1 | 1 | 0 | 2 | 2 |
| 2 | 2 | 2 | 0 | 0 |
| 3 | 3 | 3 | 1 | 0 |

Table 1.

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 2 | 2 |
| 1 | 1 | 0 | 3 | 3 |
| 2 | 2 | 2 | 0 | 0 |
| 3 | 3 | 3 | 1 | 0 |

Table 2.

They are proper, because in both cases $(3 * 2) * 1 \neq (3 * 1) * 2$.

A list of all non-isomorphic proper weak BCC-algebras having 5 elements is presented in [26]. From results proved in [9] and [26] we can deduce that for every $n \geq 5$ one can find at least 22 non-isomorphic proper weak BCC-algebras having n elements.

Any weak BCC-algebra can be considered as a partially ordered set with the partial order \leq defined by

$$(1) \quad x \leq y \iff x * y = 0.$$

From (i) it follows that in each weak BCC-algebra the implications

$$(2) \quad x \leq y \implies x * z \leq y * z$$

$$(3) \quad x \leq y \implies z * y \leq z * x$$

are satisfied for all $x, y, z \in X$.

The set of all minimal (with respect to \leq) elements of X will be denoted by $I(X)$. This is characterized in [12] and [15].

An important role in our investigations will be played by two subsets of X , namely:

$$A(b) = \{x \in X \mid x \leq b\} \quad \text{and} \quad B(a) = \{x \in X \mid a \leq x\},$$

where $a \in I(X)$. The first subset is called the *initial part* determined by $b \in X$, the second – the *branch* initiated by a .

A BCC-algebra has only one branch. Weak BCC-algebras defined by Tables 1 and 2 have two minimal elements and two branches. Namely: $I(X) = \{0, 2\}$, $B(0) = \{0, 1\}$, $B(2) = \{2, 3\}$.

Branches initiated by different elements are disjoint (cf. [12]). A weak BCC-algebra is a set-theoretic union of branches. Comparable elements are in the same branch, but there are weak BCC-algebras containing branches in which not all elements are comparable. Moreover, as it is proved in [12], elements x and y are in the same branch if and only if $x * y \in B(0)$.

In theory of BCI-algebras an important role is played by BCI-algebras satisfying some additional identities since such BCI-algebras have properties similar to properties of lattices. For example, BCI-algebras satisfying the identity

$$(4) \quad x * (x * y) = y * (y * x),$$

have properties similar to properties of lower semilattices (cf. [13]). A BCI-algebra in which

$$(5) \quad (x * y) * y = x * y$$

holds for all its elements has only one branch, so it is a BCK-algebra. The class of BCK-algebras satisfying (5) is a variety (cf. [17]). Bounded BCK-algebras satisfying the identity

$$(6) \quad x * (y * x) = x$$

are distributive lattices (cf. [17]). For BCI-algebras this is not true. In connection with this fact, M. A. Chaudhry initiated in [2], [3] and [4] the investigation of BCI-algebras in which the above identities are satisfied only by elements belonging to the same branch. The obtained results are similar, but not identical, to results proved earlier for BCI-algebras. Unfortunately, these results can not be transferred to weak BCC-algebras because in the proofs of these results the identity $(x*y)*z = (x*z)*y$ is used. But this identity holds in a weak BCC-algebra only in the case when a weak BCC-algebra is a BCI-algebra.

However, there are proper weak BCC-algebras in which the above identities are satisfied by elements belonging to the same branch.

Keeping the terminology used in the theory of BCI/BCK-algebras we will say that a weak BCC-algebra is *commutative* if it satisfies (4), *positive implicative* – if it satisfies (5), and *implicative* – if it satisfies (6). If these equations are satisfied by elements belonging to the same branch, then we will say that the corresponding weak BCC-algebra is *branchwise commutative* (*branchwise positive implicative* or *branchwise implicative*, respectively).

Example 2.2. The following proper BCC-algebra (calculated in [8])

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 2 | 2 | 2 | 0 | 1 |
| 3 | 3 | 3 | 3 | 0 |

Table 3.

is an example of a BCC-algebra which is positive implicative but it is neither commutative nor implicative. \square

Example 2.3. The weak BCC-algebra defined by the following table:

| * | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 4 | 4 | 2 | 2 |
| 1 | 1 | 0 | 4 | 4 | 2 | 2 |
| 2 | 2 | 2 | 0 | 0 | 4 | 4 |
| 3 | 3 | 2 | 1 | 0 | 4 | 4 |
| 4 | 4 | 4 | 2 | 2 | 0 | 0 |
| 5 | 5 | 4 | 3 | 3 | 1 | 0 |

Table 4.

is proper because $(5 * 3) * 2 \neq (5 * 2) * 3$. It has three branches: $B(0)$, $B(2)$, $B(4)$. Direct computation shows that it is branchwise commutative and branchwise implicative. Since $1 * (1 * 2) \neq 2 * (2 * 1)$ and $1 * (2 * 1) = 4$ it is neither commutative nor implicative. \square

3. SOLID WEAK BCC-ALGEBRAS

A crucial role in the theory of BCI-algebras is played by the equation

$$(7) \quad (x * y) * z = (x * z) * y,$$

which is valid for all $x, y, z \in X$. It is used in the proofs of many basic facts. But in many proofs it is applied only to elements belonging to some subsets. This motivates us to start studying the BCC-algebras in which (7) is fulfilled in the case where at least two elements in this equation belong to the same branch.

Definition 3.1. A weak BCC-algebra X is called *left solid* (shortly: *solid*) if the above equation is valid for all x, y belonging to the same branch and arbitrary $z \in X$. If it is valid for y, z belonging to the same branch and arbitrary $x \in X$, then we say that a weak BCC-algebra X is *right solid*. A left and right solid weak BCC-algebra is called *supersolid*.

Obviously, BCI-algebras and BCK-algebras are supersolid weak BCC-algebras. A solid BCC-algebra is a BCK-algebra since it has only one branch. There are solid weak BCC-algebras which are not BCI-algebras. For example, proper weak BCC-algebra defined by Tables 1 and 2 are not solid, but the first is right solid, the second is not right solid. A weak BCC-algebra defined by Table 4 is solid, but it is not right solid. It is the smallest solid weak BCC-algebra because proper weak BCC-algebras with 5 elements (calculated in [26]) are not solid.

Lemma 3.2. *In a solid weak BCC-algebra X we have*

$$x * (x * y) \leq y$$

for all x, y from the same branch.

Proof. Indeed, according to the definition, for x, y from the same branch we have

$$(x * (x * y)) * y = (x * y) * (x * y) = 0,$$

which proves $x * (x * y) \leq y$. □

Corollary 3.3. *In a solid weak BCC-algebra $x, y \in B(a)$ implies $x * (x * y), y * (y * x) \in B(a)$.*

Proof. Indeed, comparable elements are in the same branch (cf. [12]). □

Corollary 3.4. *In a solid weak BCC-algebra X for all $a \in I(X)$ and $x \in B(a)$ we have $x * (x * a) = a$.*

Lemma 3.5. *In a solid weak BCC-algebra for elements belonging to the same branch the following identity is satisfied:*

$$x * (x * (x * y)) = x * y.$$

Proof. Let $x, y \in B(a)$ for some $a \in I(X)$. Then $x * (x * y) \leq y$, by Lemma 3.2. This implies $x * (x * y) \in B(a)$ because comparable elements belong to the same branch. Moreover, by (3), we also have $x * y \leq x * (x * (x * y))$.

Since X is solid

$$(x * (x * (x * y))) * (x * y) = (x * (x * y)) * (x * (x * y)) = 0.$$

Thus $x * (x * (x * y)) \leq x * y$. This completes the proof. □

Theorem 3.6. *For a solid weak BCC-algebra X the following conditions are equivalent:*

- (a) X is branchwise commutative,
- (b) $x * y = x * (y * (y * x))$ for x, y from the same branch,
- (c) $x = y * (y * x)$ for $x \leq y$,
- (d) $x * (x * y) = y * (y * (x * (x * y)))$ for x, y from the same branch.

Proof. (a) \implies (b) Let $x, y \in B(a)$ for some $a \in I(X)$. Then, by Corollary 3.3, elements $y * (y * x)$ and $x * (x * y)$ are in $B(a)$. Thus

$$(x * (y * (y * x))) * (x * y) = (x * (x * y)) * (y * (y * x)) = 0.$$

Hence $x * (y * (y * x)) \leq x * y$.

Since in view of (4)

$$(8) \quad x * (x * (x * (x * y))) = (x * (x * y)) * ((x * (x * y)) * x),$$

we also have

$$\begin{aligned} (x * y) * (x * (y * (y * x))) &= (x * y) * (x * (x * (x * y))) \\ &= (x * (x * (x * (x * y)))) * y \\ &\stackrel{(7)}{=} \{(x * (x * y)) * ((x * (x * y)) * x)\} * y \\ &= \{(x * (x * y)) * ((x * x) * (x * y))\} * y \\ &= \{(x * (x * y)) * (0 * (x * y))\} * y \\ &= \{(x * (x * y)) * 0\} * y \\ &= (x * (x * y)) * y = (x * y) * (x * y) = 0, \end{aligned}$$

because $x * y \in B(0)$. Thus $x * y \leq x * (y * (y * x))$. This together with the previous inequality gives (b).

(b) \implies (c) For $x \leq y$, from (b), we obtain $0 = x * (y * (y * x))$. On the other hand, for all x, y from the same branch we have $(y * (y * x)) * x = (y * x) * (y * x) = 0$. Thus $0 = x * (y * (y * x)) = (y * (y * x)) * x$, which, by (iv), implies (c).

(c) \implies (d) Let $x, y \in B(a)$ for some $a \in I(G)$. Then $(x * (x * y)) * y = 0$, i.e., $x * (x * y) \leq y$. This, by (c), implies (d).

(d) \implies (a) If x and y are in the same branch $B(a)$ and (d) is satisfied, then we have

$$(9) \quad (x * (x * y)) * (y * (y * x)) = (y * (y * (x * (x * y)))) * (y * (y * x)).$$

But $y * (y * x) \in B(a)$, by Lemma 3.2, hence

$$(10) \quad (y * (y * (x * (x * y)))) * (y * (y * x)) = (y * (y * (y * x))) * (y * (x * (x * y))).$$

Moreover, $y * (y * x) \in B(a)$ implies also

$$0 = (y * (y * x)) * (y * (y * x)) = (y * (y * (y * x))) * (y * x),$$

which means that $y * (y * (y * x)) \leq y * x$. Multiplying this inequality by $y * (x * (x * y))$ and using (2) we obtain

$$(11) \quad (y * (y * (y * x))) * (y * (x * (x * y))) \leq (y * x) * (y * (x * (x * y))).$$

This together with (9), (10) and (11) proves that

$$(x * (x * y)) * (y * (y * x)) \leq (y * x) * (y * (x * (x * y))).$$

Since x and y are in the same branch, in view of (d), we obtain

$$(y*x)*(y*(x*(x*y))) = (y*(y*(x*(x*y))))*x = (x*(x*y))*x = (x*x)*(x*y) = 0*(x*y).$$

So,

$$(x*(x*y))*(y*(y*x)) \leq 0*(x*y) = 0.$$

Thus

$$(x*(x*y))*(y*(y*x)) = 0.$$

This shows that a solid weak BCC-algebra satisfying (d) is branchwise commutative. \square

Theorem 3.7. *A solid weak BCC-algebra X is branchwise commutative if and only if*

- (a) *each branch of X is a semilattice with respect to the operation \wedge defined by $x \wedge y = y*(y*x)$, or equivalently,*
- (b) *$A(x) \cap A(y) = A(x \wedge y)$ for all x, y belonging to the same branch.*

Proof. Let X be a branchwise commutative solid weak BCC-algebra. Then, by Lemma 3.2, for $x, y \in B(a)$, $a \in I(X)$, we have $y \wedge x = x \wedge y = x*(x*y) \leq y$. Hence $x \wedge y \in B(a)$. Moreover, for any $p \leq q$ we have $p = p*0 = p*(p*q) = q \wedge p = p \wedge q$. Thus

$$(12) \quad p \leq q \implies p = p \wedge q.$$

We prove that $p \wedge q$ is the greatest lower bound for any $p, q \in B(a)$. For this consider an arbitrary element $z \in X$ such that $z \leq p$ and $z \leq q$. Then obviously $z \in B(a)$ and

$$z*(p \wedge q) = z*(q*(q*p)) = (z \wedge q)*(q*(q*p))$$

by (12). Thus

$$z*(p \wedge q) = (q*(q*z))*(q*(q*p)) = (q*(q*(q*p)))*(q*z)$$

because $q*(q*p) = p \wedge q \in B(a)$ for $p, q \in B(a)$. The last, by Lemma 3.5, is equal to $(q*p)*(q*z)$. Hence $z*(p \wedge q) = (q*p)*(q*z) = 0$ since $q*p \leq q*z$ for $z \leq p$. So, $z \leq p \wedge q$. This means that $p \wedge q$ is the greatest lower bound of p and q . Hence the greatest lower bound of each $p, q \in B(a)$ is in $B(a)$, so $B(a)$ is a semilattice with respect to \wedge . This proves (a).

Now, if (a) is satisfied and $z \in A(x) \cap B(y)$ for some $x, y \in B(a)$, $a \in I(X)$. Then $z \leq x$ and $z \leq y$. Thus $z \in B(a)$ and $z \leq x \wedge y$, since $x \wedge y \in B(a)$ is the greatest lower bound of x and y . Hence $z \in A(x \wedge y)$. Consequently, $A(x) \cap A(y) \subseteq A(x \wedge y)$.

On the other hand, for any $z \in A(x \wedge y)$, by Lemma 3.2, we have $z \leq x \wedge y = y*(y*x) \leq x$. Hence $z \in A(x)$. Since

$$(x \wedge y)*y = (y*(y*x))*y = (y*y)*(y*x) = 0*(y*x) = 0,$$

we have $x \wedge y \leq y$ which implies $z \in A(y)$. So, $z \in A(x) \cap A(y)$. Thus $A(x \wedge y) \subseteq A(x) \cap A(y)$ and hence $A(x \wedge y) = A(x) \cap A(y)$. This proves (b).

Finally, if (b) is satisfied, then

$$A(x \wedge y) = A(x) \cap A(y) = A(y) \cap A(x) = A(y \wedge x).$$

Hence $x \wedge y \in A(y \wedge x)$ and $y \wedge x \in A(x \wedge y)$. Therefore $y \wedge x \leq x \wedge y$ and $x \wedge y \leq y \wedge x$, which implies $x \wedge y = y \wedge x$. So, X is branchwise commutative. \square

Theorem 3.8. *A solid branchwise implicative weak BCC-algebra is branchwise commutative.*

Proof. Let $x, y \in B(a)$ for some $a \in I(X)$. Then $x * (x * y) \leq y$, by Lemma 3.2. Hence $x, y, x * (x * y) \in B(a)$. Thus

$$(13) \quad (x * (x * y)) * (y * (x * (x * y))) = x * (x * y)$$

since X is branchwise implicative. Moreover, from $x * (x * y) \leq y$ and (2), it follows

$$(x * (x * y)) * (y * (x * (x * y))) \leq y * (y * (x * (x * y))).$$

Hence, by (13) and Lemma 3.2, we obtain

$$x * (x * y) \leq y * (y * (x * (x * y))) \leq x * (x * y).$$

So, $y * (y * (x * (x * y))) = x * (x * y)$ for all $x, y \in B(a)$. Theorem 3.6 completes the proof. \square

Theorem 3.9. *A solid branchwise implicative weak BCC-algebra is branchwise positive implicative if and only if it is a commutative BCK-algebra.*

Proof. By Theorem 3.8, a solid branchwise implicative weak BCC-algebra is branchwise commutative. If it is branchwise positive implicative, then for elements from the same branch we have $x * y = (x * y) * y$, which for $x = y$ implies $0 = 0 * y$. This means that this weak BCC-algebra coincides with the branch $B(0)$. Hence, it is a commutative BCK-algebra.

Conversely, if a commutative BCK-algebra is implicative, then

$$((x * y) * y) * (x * y) = ((x * y) * (x * y)) * y = 0 * y = 0$$

and

$$(x * y) * ((x * y) * y) = y * (y * (x * y)) = y * y = 0,$$

by commutativity and implicativity. Hence

$$((x * y) * y) * (x * y) = (x * y) * ((x * y) * y) = 0,$$

which implies $(x * y) * y = x * y$. So, X is positive implicative. \square

As a consequence of the above results we obtain well-known characterization of implicative BCK-algebras presented below.

Theorem 3.10. *A BCK-algebra is implicative if and only if it is both commutative and positive implicative.*

Proof. Indeed, in an implicative BCK-algebra X we have

$$x * y = (x * y) * (y * (x * y)) = (x * y) * y$$

for all $x, y \in X$. Hence, X is positive implicative. By Theorem 3.8 it also is commutative.

On the other hand, in any commutative and positive implicative BCK-algebra X we have

$$x * (x * (y * x)) = (y * x) * ((y * x) * x) = (y * x) * (y * x) = 0$$

and

$$(x * (y * x)) * x = (x * x) * (y * x) = 0 * (y * x) = 0$$

for all $x, y \in X$. This implies $x * (y * x) = x$. So, X is implicative. \square

4. POSITIVE IMPLICATIVE WEAK BCC-ALGEBRAS

Positive implicative BCK-algebras are defined as BCK-algebras satisfying the identity (5). By putting in (5) $x = y$ we can see that in a weak BCC-algebra X we have $0 * x = 0$ for all $x \in X$. This means that $X = B(0)$. Thus a weak BCC-algebra satisfying this identity has only one branch and it is a BCC-algebra. If it is solid, then it is a BCK-algebra. Hence (branchwise) positive implicative weak BCC-algebras should be defined in another way. Positive implicative BCI-algebras are defined as BCI-algebras satisfying the identity $x * y = ((x * y) * y) * (0 * y)$. (Equivalent conditions one can find in the book [13]). Such defined positive implicative BCI-algebras have properties similar to properties of positive implicative BCK-algebras. Unfortunately, the proofs of these properties are based on the identity (7). Only a small part of these results can be transferred to solid and supersolid weak BCC-algebras. In connection with this fact we introduce a new class of weak BCC-algebras called φ -implicative weak BCC-algebras.

For this we consider the self map $\varphi(x) = 0 * x$. This map was formally introduced in [10] for BCH-algebras, but earlier was used for [5] and [6] to investigations of some classes of BCI-algebras. Later, in [12], it was used to characterization of some ideals of weak BCC-algebras. Note that in weak BCC-algebras φ^2 is an endomorphism (φ is not an endomorphism, in general). Moreover, $\varphi^2(x) \leq x$ for every $x \in X$ (see for example [12]).

Definition 4.1. We say that a weak BCC-algebra X is φ -implicative if it satisfies the identity

$$x * y = (x * y) * (y * \varphi^2(y)).$$

If this identity is valid for elements belonging to the same branch then such weak BCC-algebra is called *branchwise φ -implicative*.

Obviously, a BCK-algebra is positive implicative if and only if it is φ -implicative. For BCI-algebras it is not true.

Example 4.2. Consider the following BCI-algebra (cf. [13]):

| * | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 4 | 4 | 2 |
| 1 | 1 | 0 | 4 | 4 | 2 |
| 2 | 2 | 2 | 0 | 0 | 4 |
| 3 | 3 | 2 | 1 | 0 | 4 |
| 4 | 4 | 4 | 2 | 2 | 0 |

Table 5.

This BCI-algebra has three branches: $B(0)$, $B(2)$ and $B(4)$. Since $3 * 2 \neq ((3 * 2) * 2) * \varphi(2)$, it is not (branchwise) positive implicative, but it is φ -implicative. Indeed, the case $x \leq y$ is obvious because $\varphi^2(y) \leq y$ implies $y * \varphi^2(y) \in B(0)$. Similarly the case $y \in I(X)$. The remaining five cases can be checked by standard simple calculations. \square

Example 4.3. Using the same method as in the previous example we can verify that a solid weak BCC-algebra defined by Table 4 is branchwise φ -implicative. It is not φ -implicative since $5 * 3 \neq (5 * 3) * (3 * \varphi^2(3))$. \square

One of the classical results is: *A BCK-algebra (BCI-algebra too) is implicative if and only if it is commutative and positive implicative* (cf. [13]). For BCI-algebras

this result can be extended to the branchwise version but for weak BCC-algebras this version is not valid. For weak BCC-algebras it will be valid in the branchwise φ -implicative version. For details see [11].

5. CONCLUSIONS

Solid BCC-algebras are difficult to study due to the fact that equality $(x*y)*z = (x*z)*y$ may not be true for elements belonging to different branches. However, using the calculation method presented above we can obtain results very similar to those obtained for BCI-algebras. In some cases, like for example in the first part of proof of the Theorem 3.6, our method comes down to almost mechanical transformations. In other cases we have to continue doing the transformations and choose elements so that the result is still in the same branch. This is troublesome. In exchange the results are true for the wider class of algebras.

Further investigations of solid weak BCC-algebras are continued in [11], [20] and [21]. In the first paper a description of the φ -implicative algebras is given. In the second paper results related to the f -derivations of weak BCC-algebras are proved. It turns out that they are very similar to those proved in [23] for BCI-algebras. Results presented in the third paper show that verification of various properties of ideals can be reduced to verification of these properties in some branches which is very important for computer verification.

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